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High-order perturbation expansion: application to a quantum thermodynamical system in one space dimension

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Abstract. The recently proposed method for investigating the large-order behaviour of the perturbative expansion is applied to a quantum thermodynamical system of interacting bosons in one space dimension. The method makes essential use of the classical solutions for the total Hamiltonian and of the oscillations about them. The contribution of these oscillations is calculated by using a canonical transformation related to the inverse scattering method, the relevant equation being the nonlinear Schrödinger equation. The leading-order corrections to the Bose–Einstein distribution are obtained explicitly.

1. Introduction

Recently a good deal of work has been done on the very interesting question of the large-order behaviour of the pertubative series (Lipatov 1977a, b, Brézin *et al* 1977a, b, Parisi 1977, Brézin 1978). The interest lies in two different but related aspects. The first is the old problem of the convergence of the series (Dyson 1952, Hurst 1952), which has long been a typical problem of QED, even if one realises that in comparison with experiments the perturbative expansion works exceedingly well. Once the convergence (or most likely the non-convergence) of the series has been understood, the second aspect is the problem of summing it as well as possible to obtain information useful in situations where the first orders of the perturbative expansion are less reliable, typically the case of strong coupling constants.

We present here our analysis of a one-dimensional system of interacting bosons at fixed temperature based on the Lipatov–Brézin–Le Guillou–Zinn-Justin method of large-order estimation of the perturbative series.

The system is described by the partition function

$$Z(\lambda)=N\int \mathrm{D}\psi\mathrm{D}\psi^{+}\,\mathrm{e}^{-\mathscr{A}(\psi,\psi^{+};\lambda)},\tag{1.1}$$

where

$$\mathscr{A}(\psi,\psi^{+};\lambda)=\int_{0}^{\beta}\mathrm{d}\tau\int_{-\infty}^{+\infty}\mathrm{d}x(\psi^{+}\,\partial_{\tau}\psi+\partial\psi^{+}\,\partial_{x}\psi+\mu\psi^{+}\psi+\tfrac{1}{2}\lambda)(\psi^{+}\psi)^{2});\tag{1.2}$$

β is the inverse of the temperature, μ is the negative of the chemical potential, the mass

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of the particle is $\frac{1}{2}$†, and the functional integration goes over periodic functions $\psi(0)=\psi(\beta)$, $\psi^+(0)=\psi(\beta)$ (see e.g. Bernard 1974). We have found this system worthy of investigation, since the 'instanton' structure, which plays a relevant role in the large-order estimation, is very well known, and, moreover, we have at our disposal the whole technique of the inverse scattering method, by means of which it is relatively easy to compute the quantum fluctuations around the classicial solutions. The system is also interesting in itself as a non-trivial thermodynamical system, and because we know an interesting property, clarified by Yang and Yang (1969), about the large-coupling-constant limit of the model (which yields the partition function of a free Fermi gas). In view of that, our analysis can also be considered as a test of the kind of information which can be extracted from a knowledge of the large-order behaviour of the perturbative series. We find that, while this knowledge gives information on the nature of the singularity for zero coupling constant, it is unable in itself to predict, for instance, the behaviour for large coupling constants. This behaviour is in general determined in an essential way by the terms which, from the point of view of the large-order expansion, are non-leading. We give an explicit numerical example, related to our model, in which this penenomenon is explicitly seen.

Our paper therefore has the purpose of being a non-trivial application of the method to an interesting system, and an illustration of the kind of problems it can help to understand.

2. Zero space dimensions

We begin, as an introduction, with a very simple example—the case in which our system is considered in zero dimensions. Let ψ and ψ^+ be the boson destruction and creation operators, with $[\psi, \psi^+]=1$, and let the Hamiltonian be

$$H=\mu\psi^+\psi+f(\psi^+\psi)^2.$$

The partition function for the system is

$$Z=\sum_n \mathrm{e}^{-\beta(n\mu+n^2 f)}, \tag{2.1}$$

using the representation in which $\psi^+\psi$ is diagonal and equals n. Consider now the series expansion in f:

$$Z=\sum_K (-f\beta)^K\left(\frac{1}{K!}\sum_n (n^2)^K \mathrm{e}^{-\beta\mu n}\right)=\sum_K f^K Z_K.$$

The series is divergent, since for large K

$$Z_K=\frac{\beta^K}{K!}\sum_n (-n^2)^K \mathrm{e}^{-\beta\mu n}\sim\frac{1}{\beta\mu}\left(-\frac{4}{\beta\mu^2}\right)^K\frac{K!}{(\pi K)^{1/2}}. \tag{2.2}$$

We would like now to show how the same result can be obtained by the saddle-point method in the path integral formulation, since it is the method that we will follow in the more interesting case of one space dimension.

† In the following μ will sometimes be referred to as 'mass'.

The path integral representation of Z is obtained by considering the functional integration over $\psi(\tau)$ and $\psi^+(\tau)$, with ψ and ψ^+ periodic, i.e. $\psi(0)=\psi(\beta)$, etc:

$$Z=N\int \mathrm{D}\psi(\tau)\mathrm{D}\psi^+(\tau)\exp\left(-\int_0^\beta \mathrm{d}\tau(\psi^+\partial_\tau\psi+\mu\psi^+\psi+f(\psi^+\psi)^2)\right). \tag{2.3}$$

We will normalise it at the end by considering the ratio $Z/Z(f=0)$. The large-order estimation of Z_K (defined by $Z=\Sigma_K f^K Z_K$) corresponds to the saddle-point evaluation (Brézin *et al* 1977a) of the integral

$$Z_K=\frac{(-)^K}{2\pi\mathrm{i}}N\oint\frac{\mathrm{d}f}{f}\int \mathrm{D}\psi\mathrm{D}\psi^+\,\mathrm{e}^{-K\ln(-f)}\exp\left(-\int_0^\beta \mathrm{d}z(\psi^+\partial_\tau\psi+\mu\psi^+\psi+f(\psi^+\psi)^2)\right). \tag{2.4}$$

Putting $\psi=\phi/\sqrt{|f|}$, we get a saddle point for negative f,

$$\partial_\tau\phi+\mu\phi-2(\phi^+\phi)\phi=0,\qquad -\partial_\tau\phi^++\mu\phi^+-2(\phi^+\phi)\phi^+=0,\qquad f=-(1/K)V, \tag{2.5}$$

where we have defined

$$V=\int_0^\beta \mathrm{d}\tau(\phi^+\phi)^2.$$

Let us postpone for a moment the actual solution of these equations, and let us consider the method in general. On the solution,

$$\int_0^\beta \mathrm{d}\tau(\psi^+\partial_\tau\psi+\mu\psi^+\psi+f(\psi^+\psi)^2)=-\frac{1}{f}V=K.$$

After performing the trivial integration over the oscillations in f around the saddle-point value, one finds

$$Z_K=N(-)^{K+1}(K!/K)V^{-K}(1/2\pi\mathrm{i})T_{\mathrm{osc}}, \tag{2.6}$$

where T_{osc} represents the integral, to be evaluated, over the oscillations in ψ and ψ^+.

It is clear then that the dominant contribution comes from the saddle-point configurations in which V is (in modulus) a minimum.

The solution of equations (2.5) gives

$$\phi=c\,\mathrm{e}^{-(\mu-2B)\tau},\qquad \phi^+=c^+\,\mathrm{e}^{(\mu-2B)\tau}, \tag{2.7}$$

with $B=c^+c$, and the periodicity requirement gives the constraint

$$\mu-2B=(2i\pi/\beta)l,\qquad l=0,\pm1,\pm2,\ldots.$$

Then

$$V=\beta B^2=\beta(\mu/2-\mathrm{i}\pi l/\beta)^2.$$

In general we find solutions which occur in complex conjugate pairs. Actually the dominant contribution for $K\to\infty$ corresponds to $l=0$, where the solution is real. This indeed gives the minimum value for $|V|$.

The quantity T_{osc} previously introduced in (2.6) is obtained by expanding the field around the 'classical' solution,

$$\psi(\tau)=(\mu/2|f|)^{1/2}+\psi'(\tau),$$

and integrating over ψ'. Actually it is better to work in Fourier space, and by defining

$$u_n + \mathrm{i}v_n = \frac{1}{\sqrt{\beta}} \int_0^\beta \mathrm{d}\tau\, \mathrm{e}^{-\mathrm{i}\omega_n \tau} \psi'(\tau),$$

where $\omega_n = (2\pi/\beta)n$, we find

$$T_{\mathrm{osc}} = \left(\prod_{n\neq 0} \frac{\pi}{\mathrm{i}\omega_n}\right) \int \mathrm{d}u_0\, \mathrm{d}v_0\, \mathrm{e}^{2\mu u_0^2} = \left(\prod_{n\neq 0} \frac{\pi}{\mathrm{i}\omega_n}\right) \frac{1}{\mathrm{i}} \left(\frac{\pi}{2\mu}\right)^{1/2} \int \mathrm{d}v_0.$$

It is easy to understand the meaning of the last integration: the model is actually invariant for a global phase rotation of ψ; therefore the classical solution is more generally

$$\psi_{\mathrm{cl}} = (\mu/2|f|)^{1/2}\, \mathrm{e}^{\mathrm{i}\alpha},$$

and then

$$\int \mathrm{d}v_0 = (\beta\mu/2|f|)^{1/2} \int \mathrm{d}\alpha = 2\pi(\beta\mu/2\gamma|f|)^{1/2} = 2\pi(\beta\mu/2V)^{1/2}\sqrt{K}.$$

We next evaluate for normalisation purposes the quantity $Z(f=0)$, which we find to be

$$Z(f=0) = N\frac{\pi}{\mu} \prod_{n\neq 0} \left(\frac{\pi}{\mathrm{i}\omega_n + \mu}\right),$$

so that finally we obtain

$$\frac{Z_K}{Z(f=0)} = \left(-\frac{4}{\beta\mu^2}\right)^K \frac{K!}{\sqrt{K}} \frac{1}{\sqrt{\pi}} \prod_{n\neq 0} \frac{\mu + \mathrm{i}\omega_n}{\mathrm{i}\omega_n} \tag{2.8}$$

The last infinite product is ambiguous, and we have to give a meaning to it. This is best discussed by rewriting it in the form $\exp L$, where

$$L = \sum_{n\neq 0} [\ln(\mu + \mathrm{i}\omega_n) - \ln(\mathrm{i}\omega_n)].$$

We see that the terms of the sum behave for large n as $\sim \mu/\mathrm{i}\omega_n$, and since ω_n is linear in n, the sum is formally logarithmically divergent; however, if for instance once considers the symmetric expression $\Sigma_{n>0} \ln[(\mu^2 + \omega_n^2)/\omega_n^2]$, which is formally equivalent to L, one finds a finite result. Actually the root of the ambiguity is to be found in a one-loop term which is isolated by writing

$$\begin{aligned} L &= \sum_{n\neq 0} [\ln(\mu + \mathrm{i}\omega_n) - \ln(\mu + \mathrm{i}\omega_n - \mu)] \\ &= \sum_{u\neq 0} \left(\ln(\mu + \mathrm{i}\omega_n) - \ln(\mu + \mathrm{i}\omega_n) + \frac{\mu}{\mathrm{i}\omega_n + \mu} + \frac{1}{2}\frac{\mu^2}{(\mathrm{i}\omega_n + \mu)^2} + \dots\right). \end{aligned} \tag{2.9}$$

The ambiguous term is $\mu\, \Sigma_{n\neq 0} [1/(\mathrm{i}\omega_n + \mu)]$, which is proportional to the expectation value $\langle\psi^+(0)\psi(0)\rangle$. We know (Fetter and Walecka 1971) that the correct way of treating it is to introduce the τ-ordering T_τ and define it to be $\lim_{\tau\to 0^+}\langle T_\tau(\psi^+(\tau)\psi(0))\rangle$. We then find

$$\mu \sum_{n\neq 0} \frac{1}{\mathrm{i}\omega_n + \mu} = \frac{\beta\mu}{\mathrm{e}^{\beta\mu} - 1}.$$

With this recipe we can compute L by correcting the symmetric expression

$$L=\sum_{n>0}\ln\left(1+\frac{\mu^2}{\omega_n^2}\right)-\left(\sum_{\mu>0}\frac{2\mu}{\mu^2+\omega_n^2}+\frac{1}{\mu}\right)+\frac{e^{\beta\mu}}{e^{\beta\mu}-1}=\ln\left(\frac{\sinh\mu\beta/2}{\mu\beta/2}\right)-\frac{\mu\beta}{2},$$

so that we finally get

$$\frac{Z_K}{Z(f=0)}=\left(-\frac{4}{\beta\mu^2}\right)^K\frac{K!}{(\pi K)^{1/2}}\frac{1-e^{-\mu\beta}}{\mu\beta}, \tag{2.11}$$

which coincides with the exact expression we have seen previously (see equation (2.2) and $Z(f=0)=1/(1-e^{-\mu\beta})$).

An important point which emerges from this example is the fact that for large values of K the estimation is determined by the configurations in which the 'number operator' $\psi^+\psi$ takes large values, and we know that in this case the semiclassical approximation, on which the method is based, gives sensible results.

We can make another more technical remark: the result of the asymptotic expansion turns out to be correct even though we neglected the contribution of infinitely many other saddle points that give non-leading contributions in $e^{-\mathscr{A}_{cl}}$. We will do the same in the following more complicated case.

3. One space dimension

3.1. Saddle points and classical solutions

The evaluation of the large orders in the series expansion of the partition function for the one-dimensional model follows the same pattern that we have seen in the previous section for the zero-dimensional case. The main difference occurs in the computation of the contribution of the oscillations around the classical solution, which is now much more difficult. It is convenient here to take advantage of the techniques of the inverse scattering method which have been developed to deal with the system we are considering. Essentially, we will perform a change of variables in the functional integration, and thus we will obtain the result rather straightforwardly.

We will consider the large-order expansion of the Fourier transform with respect to the space of the two-point correlation function $G^{(2)}$ taken for $\tau_1=\tau_2$, since this quantity is of more immediate interest, giving the mean value of the occupation number in momentum space, and we can eventually compare the result with the usual Bose–Einstein distribution which is obtained for the coupling constant $\lambda=0$. The starting point is the same as in § 2, namely we write

$$G^{(2)}(x_1,\tau_1;x_2,\tau_2)=\sum_K\lambda^K G_K^{(2)}$$

and

$$G_K^{(2)}=\frac{(-)^K}{2\pi i}\oint\frac{d\lambda}{\lambda}\int D\psi D\psi^+\,e^{-K\ln(-\lambda)-\mathscr{A}(\psi,\psi^+;\lambda)}\frac{\psi^+(x_1\tau_1)\psi(x_2\tau_2)}{Z(\lambda=0)} \tag{3.1}$$

where $\mathscr{A}(\psi,\psi^+;\lambda)$ is defined in equation (1.2).

Again, the saddle point corresponds to a negative value for λ, and putting for convenience $q=(-\lambda/2)^{1/2}\psi$ we obtain the following equations for the saddle-point configuration:

$$\frac{\lambda K}{2} + \mathscr{A}(q, q^{+}; -2) = 0, \tag{3.2}$$

$$\partial_\tau q - \partial_x^2 q + \mu q - 2(q^{+}q)q = 0, \qquad -\partial_\tau q^{+} - \partial_x^2 q^{+} + \mu q^{+} - 2(q^{+}q)q^{+} = 0.$$

The second and third equations are the classicial field equations known as the non-linear Schrödinger equations, which were studied by Zakharov and Shabat (1971) in their well-known work on the inverse scattering method. Notice that the coupling constant appearing here is negative, as in the Zakharov and Shabat paper, and it is therefore opposite in sign to the physical coupling constant which appears in equation (1.2). The only difference with respect to Zakharov and Shabat is that our 'time variable' τ corresponds to the analytic continuation of their time variable t, according to the substitution $t \to -\mathrm{i}\tau$, which usually occurs in the transition to statistical mechanics.

If we call $q_{\mathrm{cl}}(x, \tau), \overset{+}{q}_{\mathrm{cl}}(x, \tau)$ the solutions of equation (3.2), and put $A_{\mathrm{cl}} = \mathscr{A}(q_{\mathrm{cl}}, \overset{+}{q}_{\mathrm{cl}}; -2)$, we obtain, as in § 2, after integrating over the oscillations in λ,

$$G_K^{(2)} = (-)^{K+1} K!(2A_{\mathrm{cl}})^{-K-1} \frac{1}{2\pi\mathrm{i}} D_{\mathrm{osc}} \overset{+}{q}_{\mathrm{cl}}(x_1\tau_1) q_{\mathrm{cl}}(x_2\tau_2), \tag{3.3}$$

where the classical solutions $q_{\mathrm{cl}}(x, \tau), \overset{+}{q}_{\mathrm{cl}}(x, \tau)$ obtained from equation (3.2) are explicitly given as

$$q_{\mathrm{cl}}(x, \tau) = z\eta \frac{\mathrm{e}^{-2\mathrm{i}\xi x - (2\pi\mathrm{i}/\beta) l_2 \tau}}{\cosh[2\eta x - (2\pi\mathrm{i}/\beta) l_1 \tau]} \qquad \overset{+}{q}_{\mathrm{cl}}(x, \tau) = 2\eta \frac{\mathrm{e}^{2\mathrm{i}\xi x + (2\pi\mathrm{i}/\beta) l_2 \tau}}{\cosh[2\eta x - (2\pi\mathrm{i}/\beta) l_1 \tau]} \tag{3.4}$$

and D_{osc} represents the contribution of the integral over the oscillations of the field around the classical solution, divided by the partition function at $\lambda = 0$.

3.2 *Canonical transformation from the inverse scattering method*

In order to compute D_{osc} we turn to the inverse scattering method. First of all, we review some essential points of this method. The field variables q and q^{+} are (at given, fixed time) canonically conjugate variables in the sense of classical mechanics; therefore we define the classical Poisson bracket to be

$$[q(x), q^{+}(x')]_{\mathrm{PB}} = \delta(x, x'). \tag{3.5}$$

The inverse scattering method corresponds to a canonical transformation (Zakharov and Manakov 1974). The new set of variables are in part indexed by a discrete parameter n and in part by a continuous parameter s (these parameters replace the parameter x appearing in $q(x)$): $\xi_n, \rho_n, u_n, v_n; P(s), Q(s)$. In principle, every one of these variables takes real values from $-\infty$ to ∞. They are defined by means of an associated scattering problem, which we will not review here, and their Poisson brackets can be computed with the result

$$[\xi_n, \rho_{n'}]_{\mathrm{PB}} = \tfrac{1}{2}\mathrm{i}\delta_{n,n'}, \qquad [v_n, u_{n'}]_{\mathrm{PB}} = \tfrac{1}{4}\mathrm{i}\delta_{n,n'}, \qquad [Q(s), P(s')]_{\mathrm{PB}} = \mathrm{i}\delta(s - s'), \tag{3.6}$$

every other combination giving zero. Therefore the transformation is canonical, modulo some trivial rescaling. We also introduce

$$a_n = u_n + \mathrm{i}v_n, \qquad a_n^+ = u_n - \mathrm{i}v_n,$$
$$A(s) = (Q(s) + \mathrm{i}P(s))/\sqrt{2}, \qquad A^+(s) = (Q(s) - \mathrm{i}P(s))/\sqrt{2}. \tag{3.7}$$

such that

$$[a_n, a_{n'}^+]_{\mathrm{PB}} = -\tfrac{1}{2}\delta_{nn'}, \qquad [A(s), A^+(s')]_{\mathrm{PB}} = \delta(s-s').$$

The variables introduced by Zakharov and Shabat, namely the 'soliton' parameters $z_n = \xi_n + \mathrm{i}\eta_n$, c_n^2 and the 'continuum' parameter $\arg b(s)$, $a^2(s)$ are simply related to our variables by

$$\xi_n = \xi_n, \qquad \ln c_n^2 = \rho_n + \mathrm{i}\phi_n, \qquad a_n = \sqrt{\eta_n}\,\mathrm{e}^{\mathrm{i}\phi_n}, \qquad a_n^+ = \sqrt{\eta_n}\,\mathrm{e}^{-\mathrm{i}\phi_n}$$

and

$$A(s) = (\ln a^2(s)/\pi)^{1/2}\,\mathrm{e}^{-\mathrm{i}\arg b(s)}, \qquad A^+(s) = (\ln a^2(s)/\pi)^{1/2}\,\mathrm{e}^{\mathrm{i}\arg b(s)}.$$

The important point is that the Hamiltonian has a simple expression in terms of the new variables:

$$H = \int \mathrm{d}x[\partial_x q^+(x)\,\partial_x q(x) + \mu q^+(x)q(x) - (q^+(x)q(x))^2]$$
$$= \sum_{n=1}^{N} 4a_n^+a_n[\mu - \tfrac{4}{3}(a_n^+a_n)^2 + 4\xi_n^2] + \int_{-\infty}^{\infty} \mathrm{d}s(\mu + 4s^2)A^+(s)A(s).$$

The integer N characterises the sector in which there are N solitons.

Accordingly, the action $\mathscr{A}(q, q^+; -2)$ can be written as

$$\mathscr{A}(q,q^+;-2) = \int_0^\beta \mathrm{d}\tau\bigg(\sum_{n=1}^{N}(2i\xi_n\,\partial_\tau\rho_n - 2a_n^+\,\partial_\tau a_n) + \int_{-\infty}^{\infty}\mathrm{d}s\,A^+(s)\,\partial_\tau A(s)$$
$$+ 4a_u^+a_n[\mu - \tfrac{4}{3}(a_n^+a_n)^2 + 4\xi_n^2] + \int_{-\infty}^{\infty}\mathrm{d}s(\mu + 4s^2)A^+(s)A(s)\bigg) \tag{3.8}$$
$$+ \mathscr{G}(\beta) - \mathscr{G}(0),$$

where $\mathscr{G}$ is the generating function of the canonical transformation (Girardello and Jengo 1977). (We will see that a knowledge of $\mathscr{G}$ is not necessary for our computation.) In terms of the new variables the classical equations of motion are

$$\partial_\tau\xi_n = 0, \qquad \partial_\tau(a_n^+a_n) = 0 \tag{3.9a}$$

(and therefore ξ_n and $\eta_n = a_n^+a_n$ are constant),

$$\partial_\tau\rho_n = 16\mathrm{i}\xi_n\eta_n, \tag{3.9b}$$
$$\partial_\tau a_n = (2\mu + 8(\xi_n^2 - \eta_n^2))a_u, \tag{3.9c}$$

and, for the continuum variables,

$$-\partial_\tau A(s) = (\mu + 4s^2)A(s), \qquad \partial_\tau A^+(s) = (\mu + 4s^2)A^+(s). \tag{3.9d}$$

We are looking for periodic solutions in τ for $q(x, \tau)$ and $q^+(x, \tau)$ of period equal to β. It is seen from the equations of the inverse scattering method that this is equivalent to requiring a periodicity of period β for A and A^+, and a periodicity of period $\frac{1}{2}\beta$ for

$\exp \rho, a, a^+$. Therefore A and A^+ are zero at the saddle point, and we must have, for the soliton parameters,

$$8\xi_n\eta_n = (2\pi/\beta)l_{1n}, \qquad \mu + 4(\xi_n^2 - \eta_n^2) = (2\pi i/\beta)l_{2n}, \tag{3.10}$$

where l_{1n} and l_{2n} are positive or negative or null-integer numbers.

Since we are interested in the solution giving the minimal action A_{cl}, which is the dominant term for large K, we limit ourselves to the 'one-soliton solution', and therefore we take $N = 1$. Then the parameter n is $n = 1$ and we drop it from the following formulae.

Notice that, in the spirit of the saddle-point method, we are in general considering solutions in the analytic extension of the dynamical variables Re q and Im q. Therefore in general in our solutions q and q^+ are not complex conjugates of each other.

Actually the classical action A_{cl} is infinite for $l_1 \neq 0$, owing to the zeros of the denominator in the integration domain for x and τ, and therefore we take $l_1 = 0$, which implies $\xi\eta = 0$. Our choice is $\xi = 0$, since $\eta = 0$ would again give $A_{cl} = \infty$.

We can now compute A_{cl} for the other cases, and we find, from equation (1.2)†,

$$A_{cl} = \tfrac{32}{3}\beta\eta^3. \tag{3.11}$$

Since $\eta^2 = \mu/4 - (\pi i/2\beta)l_2$, it is clear that $|A_{cl}|$ is a minimum for $l_2 = 0$. Therefore the dominant contribution for $K \to \infty$ comes from this case, which corresponds to a static soliton solution.

The next step is the evaluation of the oscillations around this solution, that is around‡

$$a_{cl} = a_{cl}^+ = \tfrac{1}{2}\sqrt{\mu}, \qquad \rho_{cl} = 0, \qquad \xi_{cl} = 0, \qquad A_{cl} = A_{cl}^+ = 0. \tag{3.12}$$

The change of variables in the functional integration, from the old ones $q(x, \tau)$ and $q^+(x, \tau)$ to the new ones, being a canonical transformation, gives a Jacobian equal to unity. There are subtle anomalies with respect to this expected $J = 1$, which, however, only show up at the multi-loop level—as has been explicitly studied for the case of point canonical transformations (Gervais and Jevicki 1976, Salomonson 1977)—and should therefore not affect our one-loop calculation. As a check, a particular contribution, that from the oscillations independent of τ, has been calculated directly, i.e. without passing through the canonical transformation, and found to be identical with the one calculated in the way described here.

We therefore have to compute the second derivative of the action as it is written in equation (3.8) with respect to the variables $a, a^+, \rho, \xi, A, A^+$. The second derivative of $\mathscr{G}$, evaluated from the classical solution, is zero. This is so because, if we consider in general a canonical transformation from (p, q) to (P, Q), the generating function is a function, for example, of the form $\mathscr{G}(q, Q)$ and $\partial\mathscr{G}/\partial q|_Q = p$, $\partial\mathscr{G}/\partial Q|_q = P$. Now when we consider $\partial\mathscr{G}/\partial Q$ at fixed P or $\partial\mathscr{G}/\partial P$ at fixed Q, we have to think of $\mathscr{G}(q(P, Q), Q)$. It is clear, therefore, that the second derivatives of $\mathscr{G}$ with respect to P and/or Q can be expressed in terms of the old variables and derivatives of them with respect to the new ones; all these terms are, in the classical solution, periodic in τ with period β, and therefore the difference between the contributions at β and at zero vanishes.

In conclusion, we can forget $\mathscr{G}$ in computing D_{osc}.

† We cannot compute it from equation (3) since we do not know the value of $\mathscr{G}(\beta) - \mathscr{G}(0)$.

‡ Of course, a more general solution is $a = \frac{1}{2}\sqrt{\mu}\, e^{i\alpha}$, $a^+ = \frac{1}{2}\sqrt{\mu}\, e^{-i\alpha}$, $\rho = \rho_0$, with α and ρ_0 arbitrary constants. We will automatically take account of these solutions in evaluating the oscillations around the values of equation (3.11).

3.3 Computation of the integral over the oscillations

Since the action written in equation (3.8) is a sum of a part containing only the soliton variables and a part containing only the continuum variables, the integral over the oscillations factorises. Let us consider first the contribution of the soliton variables. Putting $u = u_{\rm cl} + u'$, with $u_{\rm cl} = \frac{1}{2}\sqrt{\mu}$, every other variable being zero on the saddle point, we have to compute the functional integral over periodic functions

$$D_{\rm osc}(\text{soliton}) = N \int \mathrm{D}u'\mathrm{D}v\mathrm{D}\rho\mathrm{D}\xi \exp\left(-2\int_0^\beta \mathrm{d}\tau\{2\mathrm{i}v\partial_\tau u' + \mathrm{i}\xi\partial_\tau\rho + 4\sqrt{\mu}\xi^2 - 8\mu u'^2\}\right). \tag{3.13}$$

Here the functional integral is originally over the variables $\psi = (-2/\lambda)^{1/2}q$, etc. The variables which we use are then obtained by canonically transforming ψ and ψ^+. We change variables, going to the Fourier series transforms

$$u'_m = \frac{1}{\sqrt{\beta}}\int_0^\beta \mathrm{d}\tau\, \mathrm{e}^{\mathrm{i}\omega_m\tau}u'(\tau), \qquad \omega_m = \frac{2\pi}{\beta}m,$$

and similarly for the other variables, and we choose the normalisation factor N in such a way that the integral is

$$\int \prod_m \mathrm{d}\left(\frac{\sqrt{2}u'_m}{(\Delta s)^{1/2}}\right)\mathrm{d}\left(\frac{\sqrt{2}v_m}{(\Delta s)^{1/2}}\right)\mathrm{d}\left(\frac{\rho_m}{(\Delta s)^{1/2}}\right)\mathrm{d}\left(\frac{\xi_m}{(\Delta s)^{1/2}}\right)\mathrm{e}^{-2(\ldots)}.$$

Notice that, for instance, u'_m is complex and $\mathrm{d}u'_m$ means $\mathrm{d}(\mathrm{Re}\, u'_m)\mathrm{d}(\mathrm{Im}\, u'_m)$. The reason for introducing the factor $1/(\Delta s)^{1/2}$ for every soliton variable and $\sqrt{2}$ for u'_m and v_m is that in this way the soliton variables becomes homogeneous with the continuous variables, namely for $\Delta s \to 0$ we have formally

$$\left[\frac{\sqrt{2}a}{(\Delta s)^{1/2}}, \frac{\sqrt{2}a^+}{(\Delta s)^{1/2}}\right] = -\delta(s-s')|_{s=s'}, \qquad \left[\frac{\xi}{(\Delta s)^{1/2}}, \frac{\rho}{(\Delta s)^{1/2}}\right] = \frac{\mathrm{i}}{2}\delta(s-s')|_{s=s'}$$

to be compared with $[A(s), A^+(s')] = \delta(s-s')$, or in terms of $P(s)$ and $Q(s)$, $[Q(s), P(s')] = \mathrm{i}\delta(s-s')$.

This is important, since we normalise our computation by taking the ratio with $Z(\lambda = 0)$, and for this free case the canonical transformation is trivial:

$$A(s,\tau) = \frac{1}{\sqrt{\pi}}\int \mathrm{d}x\, \mathrm{e}^{2\mathrm{i}sx}\psi(x,\tau),$$

$$A^+(s,\tau) = \frac{1}{\sqrt{\pi}}\int \mathrm{d}x\, \mathrm{e}^{-2\mathrm{i}sx}\psi^+(x,\tau),$$

Therefore we will always write the functional integral over the continuum variables as

$$\int \prod_m \mathrm{D}A_m(s)\ldots,$$

where

$$A_m(s) = \frac{1}{\sqrt{\beta}}\int_0^\beta A(s,\tau)\,\mathrm{e}^{\mathrm{i}\omega_m\tau}\mathrm{d}\tau,$$

etc. This will ensure the proper normalisation of the ratio $Z(\lambda)/Z(0)$. As a check, the quantity Δs must disappear in the final result. The integration is easy, and we obtain

$$D_{\text{osc}}(\text{soliton}) = \left(\prod_{m=1}^{\infty} \frac{\pi^4}{16(\Delta s)^4\omega_m^4}\right)2(-\mathrm{i})\left(\frac{\pi}{8\sqrt{\mu}}\right)^{1/2}\left(\frac{\pi}{16\mu}\right)^{1/2}\int\frac{\mathrm{d}\rho_0}{\Delta s}\int\frac{\mathrm{d}v_0}{\Delta s}. \tag{3.14}$$

The factor $(-\mathrm{i})$ comes from the negative eigenvalue mode u_0', and we have indicated the integrals over the zero modes. As in the zero-dimensional case, v_0 just represents the phase of the classical solution; to be more precise

$$\int \mathrm{d}v_0 = (-2/\lambda)^{1/2}\sqrt{\beta}(\mu/4)^{1/4}\int \mathrm{d}\alpha = 2\pi(\beta\mu^{1/2}/2)^{1/2}(-2/\lambda)^{1/2}. \tag{3.15}$$

The variable ρ_0 is proportional to the position in space x_0 of the soliton solution, i.e. in general and classical solution which we have to consider is $(-2/\lambda)^{1/2}q_{\text{cl}}(x-x_0,\tau)$, where $q_{\text{cl}}(x,\tau)$ is given by equation (3.4).

Comparing equation (3.4) with equation (3.9*b*) and equation (3.10) we see that

$$\int \mathrm{d}\rho_0 = (-2/\lambda)^{1/2}4\eta\sqrt{\beta}\int \mathrm{d}x_0 = 2(\beta\mu)^{1/2}(-2/\lambda)^{1/2}\int \mathrm{d}x_0. \tag{3.16}$$

The integral $\int \mathrm{d}x_0$ acts on the term $q_{\text{cl}}(x-x_0,\tau)q_{\text{cl}}^+(x-x_0,\tau)$ in equation (3.3). We now evaluate the contribution of the oscillations of the 'continuum' variables, and we consider the ratio $D_{\text{osc}}(\text{continuum})/Z(\lambda=0)$. It is convenient to 'discretise' also the continuum variables by writing $\int \mathrm{d}s\, A^+A = \Sigma_i\, \Delta s A^+(s_i)A(s_i)$, etc, and correspondingly the functional integral in the Fourier space with respect to τ is $\int \Pi_m \Pi_i\, \mathrm{d}A_m(s_i)/(\Delta s)^{1/2}$.

$$D_{\text{osc}}(\text{continuum}) = \prod_i\left[\prod_{m>0}\left(\frac{\pi^2}{(\Delta s)^2[4\omega_m^2+4(\mu+4s_i^2)^2]}\right)\frac{\pi}{\mu+4s_i^2}\,\frac{1}{\Delta s}\right]. \tag{3.17}$$

$Z(\lambda=0)$ is formally the same expression, as we have said, the difference being the fact that the density of states (i) is different, since for $\lambda \neq 0$ some of the variables describe the soliton and therefore have to be subtracted from the continuum variables. If we write

$$D_{\text{osc}}(\text{continuum}) = \exp\left(\sum_i f(s_i)d_\lambda(i)\right),$$

where $d_\lambda(i)$ is the density of the states for $\lambda \neq 0$ (Rajaraman 1975), then

$$\frac{D_{\text{osc}}(\text{continuum})}{Z(\lambda=0)} = \exp\left[\sum_{i_0} f(s_{i_0})\left(\frac{d_\lambda(i_0)}{d_0(i_0)}-1\right)\right]$$

and Σ_{i_0} is the sum over every state which appears for $\lambda=0$. It is easy to see, from the inverse scattering method, that

$$d_\lambda(i_0)/d_0(i_0) - 1 = (\Delta s/\pi)\Delta\phi/\Delta s,$$

ϕ being a phase shift, and for our soliton configuration we have the result

$$\Delta\phi/\Delta s = \sqrt{\mu}/(s^2+\mu/4).$$

In conclusion,

$$\frac{D_{\text{osc}}(\text{continuum})}{Z(\lambda=0)} = \exp\left\{\int\frac{\mathrm{d}s}{\pi}\,\frac{\sqrt{\mu}}{s^2+\mu/4}\left[\ln\left(\frac{\mu+4s^2}{\pi}\Delta s\right)\right.\right.$$
$$\left.\left.+\sum_{m=1}^{\infty}\ln\left(\frac{4\omega_m^2+4(\mu+4s^2)^2}{\pi^2}\Delta s^2\right)\right]\right\}.$$

The final result is obtained by multiplying equations (3.14) and (3.17), taking into account equations (3.15) and (3.16), and remembering that $-\lambda/2 = A_{\mathrm{cl}}/K$:

$$\frac{1}{2\pi\mathrm{i}}\frac{D_{0\mathrm{sc}}}{Z(\lambda=0)}$$
$$= -\left(\frac{K}{2A_{\mathrm{cl}}}\right)\frac{\beta}{2\pi}\exp\left\{\frac{2}{\pi}\int\frac{\mathrm{d}q}{q^2+1}\left[\ln\mu(1+q^2)+\sum_1^{\infty}\ln\left(1+\frac{\mu^2(q^2+1)}{\omega_m^2}\right)\right]\right\}\int\mathrm{d}x_0.$$

The exponent appearing in equation (3.18) seems to be a divergent quantity when summed over m and q. This happens because the logarithm in the exponent contains (when viewed in a perturbative expansion) the sum of all one-loop graphs. One of them, corresponding to a pure mass renormalisation effect, is ambiguous, and its evaluation has to be done in the way discussed in § 2 for the case with zero space dimensions†. The correct procedure is the same as in equation (2.9) and (2.10), and we therefore write

$$\sum_{m>0}\ln\left(1+\frac{\mu^2(q^2+1)}{\omega_m^2}\right)=\sum_{m\neq 0}\{\ln[\mu(q^2+1)+\mathrm{i}\omega_m]-\ln[\mu(q^2+1)+\mathrm{i}\omega_m-\mu(q^2+1)]\}.$$

We then expand the second logarithm in $\mu(q^2+1)$ and evaluate correctly the ambiguous term $\Sigma_{m\neq 0}\,1/[\mathrm{i}\omega_m+\mu(q^2+1)]$ according to the prescription of § 2. In this way the sum gives

$$\ln\left(\frac{\sinh(\beta\mu/2)(q^2+1)}{(\beta\mu/2)(q^2+1)}\right)-\frac{\beta\mu}{2}(q^2+1),$$

and the complete exponent is therefore

$$\frac{2}{\pi}\int\frac{\mathrm{d}q}{q^2+1}\left(\ln\mu(q^2+1)+\ln\sinh\frac{\beta\mu}{2}(q^2+1)-\ln\frac{\beta\mu}{2}(q^2+1)-\frac{\beta\mu}{2}(q^2+1)\right),$$

which is convergent in q. Substituting in equation (3.3) we find

$$G_K^{(2)}=\left(\frac{-1}{2A_{\mathrm{cl}}}\right)^K KK!\frac{g}{32\beta^2\mu^3}\frac{\beta}{4\pi}\mu^2\frac{\mathrm{e}^{F(\beta\mu)}}{(\beta\mu)^2}\int\mathrm{d}x_0\,q_{\mathrm{cl}}(x_1-x_0,\tau_1)q_{\mathrm{cl}}^{+}(x_2-x_0,\tau_2),\tag{3.19}$$

where $F(y)$ is given by

$$F(y)=\frac{2}{\pi}\int\frac{\mathrm{d}q}{q^2+1}\ln(1-\mathrm{e}^{-y(q^2+1)})=-2\sum_e\frac{d}{l}\operatorname{erfc}(ly)^{1/2}.\tag{3.20}$$

This represents the large-order expansion of the correction to the finite-temperature propagator. Taking the Fourier transform with respect to the space and 'time', we obtain the representation in terms of the usual variables p and $\omega_n=(2\pi/\beta)n$ (the integral over x_0 in equation (3.19) ensures momentum conservation). Our result can be written in the following way, with $g=3\lambda/8\beta\mu^{3/2}$:

$$G^{(2)}(p,\omega_n)=\frac{1}{\mathrm{i}\omega_n+p^2+\mu}+\ldots+\delta_{n,0}\frac{1}{\mu}\left(\sum_{K\,\mathrm{large}}(-)^K g^K KK!\right)\frac{9\pi}{64}\frac{1}{\cosh(\pi p/2\sqrt{\mu})}\frac{\mathrm{e}^{F(\beta\mu)}}{(\beta\mu)^2}.\tag{3.21}$$

† Once the correct prescription has been followed there is no need for explicit mass renormalisation and therefore no counter term of this kind will appear in the Hamiltonian (Fetter and Walecka 1971).

Two limiting cases, where the expression simplifies considerably, are

$$e^{F(y)} = 16y^2 \quad \text{as} \quad y \to 0, \qquad e^{F(y)} \to 1 \quad \text{as} \quad y \to \infty.$$

Notice again that the dominant contribution for large K only affects the mode $n = 0$.

Finally, the large-order contribution can be Borel-summed according to the formula

$$\sum_{k=1}^{\infty} (-K) g^K K! K = \frac{1}{g} e^{1/g}\left(\mathscr{E}_2\left(\frac{1}{g}\right) - \mathscr{E}_1\left(\frac{1}{g}\right)\right) \equiv s(g), \tag{3.22}$$

where $\mathscr{E}_1(z)$ and $\mathscr{E}_2(z)$ are the exponential integral functions defined, for example, in Abramowitz and Stegun (1968).

Now for small values of g this sum is not very different in actual value from the first terms of the asymptotic expansion from which it was obtained, e.g. for $g = 0{\cdot}1$ the actual value is $s(g) = -0{\cdot}08$, the first term gives $-0{\cdot}1$, and the sum of the first and second terms gives $-0{\cdot}06$.

For large g the behaviour of $s(g)$ is $\sim(1/g)(\ln g + \gamma + 1)$, where γ is the Euler constant. However, as will be discussed in § 4, the meaning of the sum for large g is not free from ambiguities.

4. Discussion of the result

In order to discuss the meaning of the result obtained in § 3 let us consider the occupation number $n(p)$ in momentum space, which is proportional to the Fourier transform, with respect to space, of $\lim_{\epsilon\to 0}\langle\psi^+(\epsilon, x)\psi(0, 0)\rangle$ (Fetter and Walecka 1971). In terms of the Green function $G^{(2)}(p, \omega_n)$ we have

$$\begin{aligned} n(p) &= \frac{1}{\beta} \lim_{\epsilon\to 0} \sum_{n=-\infty}^{\infty} e^{i\omega_n\epsilon} G^{(2)}(p, \omega_n) \\ &= \frac{1}{e^{\beta(p^2+\mu)} - 1} + \ldots + \frac{9\pi}{64}\frac{1}{\beta\mu}\frac{1}{\cosh^2(\pi p/2\sqrt{\mu})} \sum_{K\,\text{large}} (-)^K g^K K K! \frac{e^{F(\beta\mu)}}{(\beta\mu)^2}. \end{aligned} \tag{4.1}$$

From our previous discussion it is clear that we have just computed the leading terms of the correction for $K \to \infty$. It is possible to perform the Borel sum of the large-K contribution, as we have explicitly done at the end of § 3.3. Notice, however, that the actual result of the Borel sum is ambiguous, firstly because we have arbitrarily extrapolated the result for small K, and secondly because one can rewrite the term $K!K$ in different ways, which leave invariant the leading order but can modify the result; for instance, we could have used $(2\pi)^{1/2} e^{-K} K^{K+3/2}$. Indeed, the method essentially gives information on the kind of singularity one has for $g \to 0$; as we see, it is a logarithmic branch point, as is usual in theories of this type. The next question could be whether the method is also able to give information on the strong-coupling regime, that is for $g \to \infty$. One might think that this is possible, since the Borel sum, within some ambiguities, represents a way of handling the most divergent contribution.

Here we can make a rather precise test of the method, since we know from the work of Yang and Yang (see also Thacker 1977) that, in the one-dimensional case we are considering, the limit for $g \to \infty$ of $n(p)$ is a Fermi–Dirac distribution, namely

$$\lim_{g\to\infty} n(p) = \frac{1}{e^{\beta(p^2+\mu)} + 1}.$$

In fact we do not find any indication of this result in equation (4.1). Let us try, then, to understand the reason for this.

One can re-derive schematically the result of Yang and Yang (1969) by considering a lattice in x space and field variables at each lattice site $\psi(x_i)$, $\psi^+(x_i)$, with the commutation relations $[\psi(x_i), \psi^+(x_j)] = \delta_{ij}$. Writing the single-site Hamiltonian as

$$H_i = \mu\psi^+(x_i)\psi(x_i) + g\psi^{+2}(x_i)\psi^2(x_i) = (\mu - g)n_i + gn_i^2,$$

where n_i is the number operator $n_i = \psi^+(x_i)\psi(x_i)$, we see that in the limit $g \to \infty$ only the states $n_i = 0, 1$ have a finite energy. At the single-site level, then, we obtain Fermi-like operators by representing $\psi(x_i)$ and $\psi^+(x_i)$ in this subspace. In the one-dimensional lattice it is then possible, by a standard trick (Lieb *et al* 1961, Pfeuty 1970), to transform these operators into anti-commuting Fermi operators for different sites, and from that we obtain the result. It is apparent from this analysis that the dominant configuration for large g corresponds to small occupation numbers, where any semiclassical approximation would fail. On the other hand, we have already noticed that the method for obtaining the large-order estimation relies on a semiclassical approximation. It follows from these considerations that the strong-coupling features of the theory do not appear in the dominant terms for large K. The sum of these dominant terms is not very relevant in the limit $g \to \infty$.

We would like to illustrate this point by means of a simple example. Let us consider as a problem the approximate computation of the integral

$$Z(g) = \int \mathrm{d}u \; \mathrm{e}^{-gu^4 - \mu u^2 + gu^2} \equiv \int \mathrm{d}u \; \mathrm{e}^{-A(u)}. \tag{4.2}$$

This example is tailored in order to produce, in the application of the saddle-point method we are studying, two relevant saddle points, the 'field' variable u being large in one of them and of the order of unity in the other (see below). This is the reason for the term gu^2 in the exponent of the integrand: we can think of it as a kind of mass counter-term, as it is necessary for the path integral formulation of a normal-order prescription in the interaction. In the one-dimensional system we have considered in this paper there was no need for a counter-term, the normal ordering being ensured by the particular prescription used to evaluate the ambiguous infinite products, as discussed in §§ 2 and 3.

It is easily seen that

$$Z(g) \underset{g \to \infty}{\longrightarrow} (2\pi/g)^{1/2} \, \mathrm{e}^{g/4 - \mu/2}.$$

The coefficient of the asymptotic expansion in g can also be computed exactly, and we have the formal series

$$Z = \sum_K g^K Z_K$$

with

$$Z_K = \frac{(-)^K}{K!} \sum_{l=0}^{K} \binom{K}{l} \frac{(-)^l}{\mu^{2K-l+1/2}} \Gamma(2K - l + \tfrac{1}{2}). \tag{4.3}$$

Let us now apply the saddle-point method to evaluate Z_K. Following the discussion of §§ 2 and 3, we obtain that the saddle point in the variables g and u is, in the limit of large K,

$$g = g_{\mathrm{cl}} \sim -\mu^2/4K, \qquad u^2 = u_{\mathrm{cl}}^2 \sim 2K/\mu + \tfrac{1}{2}. \tag{4.4}$$

By integrating over the oscillations according to the general prescriptions, we obtain the contribution

$$Z_K^{\mathrm{cl}} \sim (-4)^K K^{K-1/2}\,\mathrm{e}^{-K}\,\mathrm{e}^{-\mu/2}(1/\mu^{2K+1/2}). \tag{4.5}$$

This is indeed the leading term for large K of the exact result written in equation (4.3). There is, however, another saddle point, for

$$g = g_{\mathrm{L}} \sim 4K, \qquad u^2 = u_{\mathrm{L}}^2 \sim \tfrac{1}{2} - \mu/8K, \tag{4.6}$$

which, again after the various integrations, gives the contribution

$$Z_K^{(\mathrm{L})} \sim (\sqrt{\pi}/2)(\tfrac{1}{2})^{2K+1/2}(1/\Gamma(K+\tfrac{3}{2}))\,\mathrm{e}^{-\mu/2}. \tag{4.7}$$

This, of course, is non-leading with respect to the previous one. However, it is precisely the series (convergent in the usual sense and non-alternating in sign)

$$\sum_K g^K Z_K^{(\mathrm{L})} = \left(\frac{2\pi}{g}\right)^{1/2} \mathrm{e}^{g/4-\mu/2} \tag{4.8}$$

which gives the result for $g \to \infty$. Here we see that the relevant configuration for large g is the one in which the 'field' u is not large. If we think of the Borel method of summation as a sort of dispersion relation in the coupling constant (Cardy 1977), where a relevant role is played by the singularities for small g, the other term we find, namely $\Sigma_K\, g^K Z_K^{(\mathrm{L})}$, corresponds to the addition of an entire function to the dispersion representation.

Our conclusion is that the method of large-K expansion plus a Borel summation prescription gives useful information on the behaviour of the theory for small coupling constants. In particular, it enables one to deal with the formally diverging perturbative series—take as an example the compact result one obtains by substituting equation (3.22) into equation (4.1). However, the other regime, namely that of large coupling constants, can in general be affected in a decisive way by terms which are negligible compared with the leading ones for large orders in g.

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